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The game chromatic index of forests of maximum degree $\Delta \geqslant 5$

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Abstract

Using a refinement of the methods of Erdös et al. [Note on the game chromatic index of trees, Theoret. Comput. Sci. 313 (2004) 371–376] we prove that the game chromatic index of forests of maximum node degree 5 is at most 6. This improves the previously known upper bound 7 for this parameter. The bound 6 is tight [P. Erdös, U. Faigle, W. Hochstättler, W. Kern, Note on the game chromatic index of trees, Theoret. Comput. Sci. 313 (2004) 371–376].

Keywords: Game chromatic index; Forest; Graph colouring games; Game chromatic number; Strongly unmatched edge

1. Introduction

Consider the following game, played on an uncoloured graph G with regard to a certain set C of colours. Two players, Alice (A) and Bob (B), move alternately. A move consists in colouring an uncoloured edge of G with a colour from C so that adjacent edges are not coloured with the same colour. The game ends when no more move is possible. Alice wins if every edge is coloured at the end of the game, otherwise Bob wins.

In order to define the game properly, we assume that Bob has the first move and Bob is allowed to miss one or several turns, whereas passing is not allowed for Alice. Weaker variants of the game are possible and will be discussed in Section 5.

The *game chromatic index* $\chi'_g(G)$ *of the graph* G is the smallest number $n \in \mathbb{N}_0$, so that Alice has a winning strategy for the game played on G with n colours.

The game chromatic index is a variation of the *game chromatic number* that is analogously defined for a game where nodes are coloured instead of edges. Games of this type were introduced by Bodlaender [2]. During the last decade interest in game chromatic numbers of certain classes of graphs has increased (see also the references in [4–6,9]).

Cai and Zhu [3] studied the game chromatic index for a game where Alice has the first move and passing is not permitted. They achieved the upper bound $\Delta + 3k - 1$ for the game chromatic index of k-degenerate graphs with maximum node degree Δ, a bound implying the upper bound $\Delta + 2$ for forests. In case of trees with an odd number of edges and maximum degree 3 they tightened this bound to the value 4. The latter result raised the question, whether it is true in general that the game chromatic index of forests with maximum degree $\Delta \in \mathbb{N}_0$ is bounded above by $\Delta + 1$. This trivially holds for $\Delta \leqslant 2$.

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A breakthrough arose from the article of Erdös et al. [6] which gives an affirmative answer in the cases $\Delta \geqslant 6$, in fact even for the game we study here. In Sections 2–4 we modify their strategy and generalize their result to the cases $\Delta \geqslant 5$, i.e. we prove

Theorem 1. *Let $\Delta \geqslant 5$ and F be a forest of maximum degree $\Delta(F) \leqslant \Delta$. Alice has a winning strategy for the game on F with $\Delta + 1$ colours, thus*

$$\chi'_g(F) \leqslant \Delta + 1.$$

Erdös et al. [6] pointed out that the best possible bound is $\Delta + 1$ for arbitrary $\Delta \geqslant 2$. With our methods, however, it is not possible to close the gap at $\Delta = 3$ and $\Delta = 4$ (cf. Section 5). Recently, the author completed the case $\Delta = 3$, using a substantially different strategy [1]. Indeed, in the article [8] of He et al. the case $\Delta = 3$ was already examined, but their proof seems to be incomplete. The case $\Delta = 4$ still remains open.

2. Outline of Alice's strategy

Let $\Delta \geqslant 5$ and F be a forest of maximum degree $\Delta(F) \leqslant \Delta$. Further, let C be a set of $\Delta + 1$ colours. We describe a winning strategy of Alice, assuming Bob has the first move and passing is allowed for Bob only. The fundamental idea of Alice's strategy, a decomposition of trees, also forms the basis of the strategies in [3,6,1]. It may be used as well to prove the result of Faigle et al. [7] that the game chromatic number of a tree (forest) is at most 4.

Formally, an *independent subtree* T is a subtree of F together with its partial colouring in a certain situation of the game, so that every coloured edge of T is a leaf edge in T and T is maximal with this property. So the uncoloured leaf edges of T are leaf edges of F as well. Unless a leaf edge is coloured, a move can be regarded as splitting an independent subtree T into two new independent subtrees, the just coloured edge belonging to both new independent subtrees, every other edge of T only occurring in one of them.

In Section 3 we will define some special classes of independent subtrees, the *permitted types*. During the game, Alice's strategy maintains the property that after each of her moves, before Bob's next move, every independent subtree is of a permitted type. Initially, this property holds.

For every independent subtree T of each permitted type that contains an uncoloured edge, we have to prove that, on the one hand, a move on T which creates only permitted types is always possible. On the other hand, we must ensure that, if Bob colours an edge in T, at most one of the resulting new independent subtrees is not permitted, and that, if there is such a forbidden independent subtree T', Alice has a feasible move on T' to reinstall her strategy. In doing so, only colours of C may be used. If this procedure is possible for T, we say that T *can be reduced to permitted types*. If every independent subtree of a certain type can be reduced to permitted types, we also say that this type *can be reduced to permitted types*. As long as there are still uncoloured edges, permitted types can be reduced to permitted types as will be shown in Section 4. By induction, Alice wins.

3. The permitted types

To describe the permitted types in detail, we start with some definitions. Let H be an independent subtree with at least three coloured edges. A *star node* is a node that lies on all paths between different coloured edges. If a star node exists in H, it is the unique star node of H. Every independent subtree with three coloured edges contains a star node.

For $0 \leqslant n \leqslant \Delta$, an independent subtree is an *n-star* if it contains exactly n coloured edges and has, in case $n \geqslant 3$, a star node. An n-star is *regular* if $n \leqslant 2$ or if it has at least one coloured edge incident with the star node.

Let v_0 be a node of an independent subtree T. A *v_0-branch* B is a subtree of T, so that exactly one edge of B is incident with v_0, and B is maximal with this property. An *uncoloured v_0-branch* is a v_0-branch that contains no coloured edge, otherwise the v_0-branch is called *coloured*. An n-star ($n \geqslant 3$) with star node v_0 contains exactly n coloured and at most $\Delta - n$ uncoloured v_0-branches.

Erdös et al. [6] defined a coloured edge in an n-star as *unmatched* if $n \geqslant 3$ and its colour is different from the colours of the edges incident with the star node (i.e. those of distance 0). However, a more subtle notion is needed. An unmatched edge is called *strongly unmatched* if it has distance 1 to the star node.

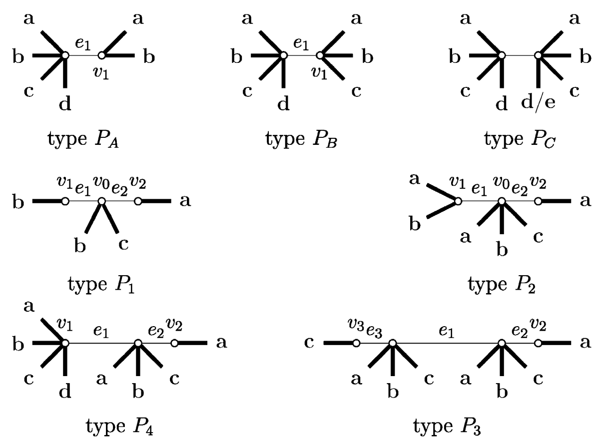

Fig. 1. The permitted types (except S_n) in the case $\Delta = 5$. Italic letters denote the names of some nodes and uncoloured edges. Thick edges are coloured. Different boldface letters symbolize different colours. Note that there may be other uncoloured subtrees incident with nodes of degree less than 5 in the figure. In particular, type P_1 has an uncoloured v_0-branch (otherwise it would be of type S_4).

Erdös et al. [6] tried to take n-stars as permitted types, but observed that unmatched edges cause a problem. So they restricted the permitted types to n-stars which contain at most

$$\max(0, 6 - 1 - n)$$

unmatched edges if the star node degree is at least 6. Hereby, they proved that Alice's strategy from Section 2 is a winning strategy if $\Delta \geqslant 6$.

However, this definition of permitted types is much too restrictive. In order to guarantee that, in case $\Delta = 6$, Bob cannot construct a Δ-star with two strongly unmatched edges of the same colour, Erdös et al. demand that every $(\Delta - 1)$-star (with full star node degree Δ) has no unmatched edges after Alice's move, which is the main idea of their strategy. But their argument applies to $\Delta > 6$, too. So we might rather consider n-stars which contain at most

$$\max(0, \Delta - 1 - n)$$

unmatched edges if the star node degree is Δ. The strategy may be weakened once more, if we replace 'unmatched' by 'strongly unmatched', but demand that the permitted types be regular. (This is no restriction, since the permitted types of Erdös et al. are always regular.)

As a part of Lemma 2, we will prove that the weakened strategy works for $\Delta \geqslant 6$. We might try to apply this weakened strategy to the case $\Delta = 5$, which means: we demand that a 3-star has at most one strongly unmatched edge after Alice's moves. Nevertheless, Alice, in general, cannot avoid producing 3-stars with two strongly unmatched edges during the game. Thus we define the permitted types S_n for $\Delta \geqslant 5$:

An n-star T is of type S_n if

(1) T is regular, and
(2) T contains at most $\max(0, \Delta - 1 - n)$ strongly unmatched edges, if $n \geqslant 4$ and the star node has full degree Δ.

The main modification of the strategy in case $\Delta = 5$ consists in allowing 3-stars to have two strongly unmatched edges. Of course, we have to pay a price for it. As we shall see in Section 4, Bob may create a (single) configuration that cannot be reduced to the permitted types mentioned so far. Therefore, in case $\Delta = 5$, we have to permit some other types of independent subtrees which are depicted in Fig. 1. By a sequence of reductions these types are decomposed to types S_n, which is subject of Lemmata 3–7.

4. Reducing the permitted types

Recall Alice's strategy: after Alice's moves all independent subtrees are permitted, where a subtree is permitted if it is of type S_n or in case $\Delta = 5$ of one of the permitted types in Fig. 1. The lemmata of this section prove that it is a winning strategy.

Lemma 2. *Let* $0 \leqslant n \leqslant \Delta$. *An independent subtree* T *of type* S_n *that contains an uncoloured edge can be reduced to permitted types.*

Proof. Obviously, a move on T creating only permitted types is possible. Whenever Bob splits T into two independent subtrees, one of the new subtrees is of type S_1 or S_2. We have to examine the cases where Bob generates an independent subtree T' that is not permitted. If T' is a 3-star, Alice can easily reinstall regularity. Thus, we may assume that T' contains at least four coloured edges. Consequently, T has a star node v_0. There are three possibilities for T' to have been created from T: (i) Bob may have coloured an edge in an uncoloured v_0-branch. Then $n \leqslant \Delta - 1$ and T' is a regular $(n+1)$-star. (ii) Bob may have coloured an edge on the unique path from a coloured edge to v_0. In this case T' is a new regular n-star. (iii) Bob may have coloured an edge e in a coloured v_0-branch that does not lie on the paths from coloured edges to v_0. Then T' is called $(n+1)$-*shooting star* with *tail node* v_1, where v_1 is the first of the nodes of all paths from coloured edges to v_0 that one meets when following the unique path from e to v_0.

Case 1: T' is a regular k-star ($k \in \{n, n+1\}$).

Bob may have created at most one additional unmatched edge. So, in general, T' has at most two strongly unmatched edges more than permitted by (2) for type S_k. In case $\Delta = 5$ and $k = 4$, however, it may have three strongly unmatched edges, whereas type S_4 permits none of them in case $\Delta = 5$. There is only one strongly unmatched edge in T', if $k = \Delta$.

Alice eliminates a single strongly unmatched edge $\tilde{e}$ and possibly every strongly unmatched edge of a second colour by colouring the edge between $\tilde{e}$ and v_0 with the second colour (if it exists; otherwise with any feasible colour). If T' has exactly two or three strongly unmatched edges of the same colour x, she colours an uncoloured edge incident with v_0 with x. (Such an edge exists, since $k < \Delta$ in that case, and T' has full star node degree Δ, otherwise T' would be permitted.) We are left to the case that $\Delta = 5$, $k = 4$ and T' has three strongly unmatched edges in distinct colours. Clearly, Alice has no chance to reduce T' to types S_m. But she may eliminate two strongly unmatched edges to create type P_1.

Note that we profit, at this step, from considering strongly unmatched rather than unmatched edges. Without this refinement of the strategy, we would be forced to additional case distinctions, as we would have to permit some more extra types (which are, by our strategy, included in some type S_m).

Case 2: T' is an $(n+1)$-shooting star.

This case is similar to the 'split move' case of Erdös et al. [6]. Let e_0 (resp. e_1) be the first (resp. last) edge on the unique path from v_0 to the tail node v_1 of T'. In every following subcase, Alice splits T' into types S_n and S_3.

Subcase 2.a: $e_0 = e_1$.

Alice colours e_0 feasibly. In case of $n = \Delta$, condition (2) for T implies that in T' the edges adjacent to e_0 are coloured in at most Δ different colours. Thus one colour is left for Alice.

Subcase 2.b: $e_0 \neq e_1$.

If there is a coloured edge incident with v_1, Alice colours e_0 different from the colours of the edges incident with v_0. Otherwise, if no coloured edge is incident with v_1, colouring e_0 would result in a non-regular 3-star. So in the latter case Alice colours e_1 with the colour of an edge incident with v_0. Condition (1) for T implies the existence of such a colour. □

Remark. Alice may be forced to create 3-stars with two strongly unmatched edges, which is allowed, but will make fail a corresponding strategy for $\Delta = 4$.

We are left to reduce the special permitted types of Fig. 1 to which we refer in the following. In most cases, Alice will reduce them to types S_n. Obviously, Alice always has a feasible move on these permitted types. If Bob colours any of the edges (e_3), e_1 or e_2 in any permitted type, a move of Alice that reinstalls her strategy by using one of the remaining e_i is easily found. Hence we have to focus on the cases when Bob is playing on an edge not shown in Fig. 1, thereby creating an independent subtree that is not permitted.

Lemma 3. *Type P_1 can be reduced to permitted types.*

Proof. If Bob colours an edge in an uncoloured v_2-branch, Alice colours e_2. (Generally, there seems to be no alternative for Alice's move. Section 5 exemplifies that Alice's strategy, if generalized to the case $\Delta = 4$, will fail at this step.)

Further, we consider the case that Bob colours an edge e in the uncoloured v_0-branch. If e is incident with v_0 then Alice colours e_2. Otherwise, whenever e is at distance at least 2 from v_0, or at distance 1 and coloured different from **a**, Alice colours the last edge on the unique path from e to v_0 with **a**. If e at distance 1 is coloured itself with **a**, she must colour e_1 with **a** in order to obtain a decomposition into permitted types.

Finally, if Bob colours an edge e in an uncoloured v_1-branch, Alice answers by colouring e_1 with **a**, unless e is incident with v_1 and coloured with **a**. In this case she colours the unique edge incident with v_0 in the uncoloured v_0-branch with **a**, creating an independent subtree of type P_2. Note that the uncoloured v_0-branch exists by definition of type P_1. $\square$

Lemma 4. *Type P_2 can be reduced to permitted types.*

Proof. If Bob colours an edge in an uncoloured v_2-branch then Alice colours e_2 to create type P_A. If Bob, with colour x, colours an edge e in an uncoloured v_1-branch, Alice tries to reinstall her strategy by colouring e_1, preferably with x, else feasibly. This only fails if e, at distance 1 from v_1, is coloured with **c** and v_1 has full degree 5. In that special case Alice creates type P_3 by colouring the edge incident with v_1 in the remaining uncoloured v_1-branch with colour **c**. There is no uncoloured v_0-branch since $\Delta = 5$. $\square$

Lemma 5. *Type P_3 can be reduced to permitted types.*

Proof. Apart from the trivial cases that Bob chooses e_3, e_1 or e_2 by his move, we may, by reasons of symmetry, assume that Bob colours an edge in an uncoloured v_3-branch. Then Alice colours e_3 feasibly and creates type P_4. $\square$

Lemma 6. *Type P_4 can be reduced to permitted types.*

Proof. If Bob colours an edge in an uncoloured v_2-branch, Alice colours e_2 and obtains type P_C. $\square$

Lemma 7. *The types P_A, P_B and P_C can be reduced to permitted types.*

Proof. The only relevant types to consider are type P_A (resp. P_B). If Bob colours an edge incident with v_1, colouring e_1 reduces the configuration to trivial stars. If he colours an edge e that is not incident with v_1, Alice, by colouring the last edge on the unique path from e to v_1, turns type P_A into type P_B (resp. P_B into P_C). (She must use either colour **c** or **d** in the case of type P_A.) $\square$

This completes the proof of Theorem 1. To illustrate Alice's strategy in the case of $\Delta = 5$ we list the two main paths that lead back to the stars:

$$S_3 \to P_1 \to P_2 \to P_3 \to P_4 \to P_C \to S_5,$$

$$S_3 \to P_1 \to P_2 \to P_A \to P_B \to P_C \to S_5.$$

Clearly, these paths may be left earlier.

5. Final remarks and comments

Remark. Other variants of the game are possible. More generally, if we specify which player $X \in \{A, B\}$ has the first move, and whether one player $Y \in \{A, B\}$ has the right to pass or none of them ($Y =$ '$-$') is allowed to miss a turn, we obtain a game $\mathcal{G}(X, Y)$. The *game chromatic index $\chi'_{[X,Y]}(G)$ for the game $\mathcal{G}(X, Y)$ of the graph G* is the smallest

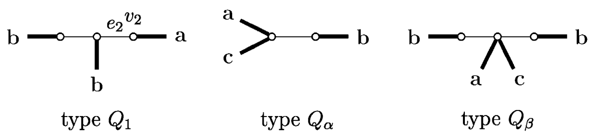

Fig. 2. The way the strategy fails for $\varDelta = 4$.

number $n \in \mathbb{N}_0$, so that Alice has a winning strategy for this game played on G with n colours. The different games and their respective game chromatic indices are closely related. It is useful to consider the game $\mathscr{G}(A, A)$ for lower and $\mathscr{G}(B, B)$ for upper bounds of game chromatic indices. Indeed, for any graph G

$$\chi'_{[A,A]}(G) \left\{ \begin{array}{cccccc} \overset{(1)}{\leqslant} & \chi'_{[A,-]}(G) & \overset{(2)}{\leqslant} & \chi'_{[A,B]}(G) & \overset{(3)}{\leqslant} \\ \overset{(4)}{\leqslant} & \chi'_{[B,A]}(G) & \overset{(5)}{\leqslant} & \chi'_{[B,-]}(G) & \overset{(6)}{\leqslant} \end{array} \right\} \chi'_{[B,B]}(G).$$

Proof. (3) and (4) are obvious. Concerning (1), (2), (5) and (6), the difference between the respective games consists in the rule whether a certain player is allowed to pass or not. Let us prove (2). If Bob has a winning strategy with n colours for $\mathscr{G}(A, -)$, he also has it for $\mathscr{G}(A, B)$, because in the second game he may make no use of his right to miss a turn, unless he is forced to pass. But this only happens if no more move is possible in $\mathscr{G}(A, -)$, so that Alice has no next move in $\mathscr{G}(A, B)$, and Bob wins in either game. The proofs of (1), (5) and (6) are similar. $\square$

Of course, these inequalities also hold for game chromatic numbers instead of indices.

Tightness of the bound. Let $X \in \{A, B\}$ and $Y \in \{A, -, B\}$. For any non-empty class $\mathscr{K}$ of graphs we define

$$\chi'_{[X,Y]}(\mathscr{K}) := \sup_{G \in \mathscr{K}} \chi'_{[X,Y]}(G).$$

Erdös et al. [6] exhibit a tree $T_\varDelta$ with $\varDelta(T_\varDelta) = \varDelta$ and $\chi'_{[X,Y]}(T_\varDelta) = \varDelta + 1$, for any $\varDelta \geqslant 2$. Combining this result with Theorem 1 and the remark above, for the class $\mathscr{F}_\varDelta$ of forests of maximum degree at most $\varDelta$, we obtain:

Corollary 8. $\chi'_{[X,Y]}(\mathscr{F}_\varDelta) = \varDelta + 1$ *for* $\varDelta \geqslant 5$.

The strategy fails for $\varDelta = 4$. The strategy, when applied to the case $\varDelta = 4$, demands that, after Alice's moves, 3- and 4-stars do not contain a strongly unmatched edge. However, it allows a single exceptional configuration Q_1 which comes from the reduction of a 3-star with three strongly unmatched edges, produced by Bob. Q_1 is depicted in Fig. 2 and corresponds to type P_1. If Bob colours an edge at distance 1 from v_2 in an uncoloured v_2-branch with **b**, Alice, according to her strategy, has to colour e_2 different from **a** and **b**. Thus she creates type Q_α which is forbidden by her strategy. Moreover, Bob may turn Q_α into Q_β (cf. Fig. 2). Q_β corresponds to the independent subtree in Fig. 3 of Erdös et al. [6] which is an example that the strategy presented in [6] fails for $\varDelta = 5$. It is easily checked that Bob has a winning strategy on Q_β for $\varDelta = 4$, if Q_β contains enough uncoloured edges. So the fact that it is impossible to avoid certain 3-stars by our strategy makes the case $\varDelta = 4$ significantly harder.

Acknowledgements

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